Numerical Solution to a Dual Mass-Spring-Damper System Using Matlab

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Introduction:

A mass-spring-damper system is a system with two blocks of mass m_1 and m_2, in which Block B_1 is connected to an immovable wall by a spring and damper and B_2 is connected to B_1 by a second spring and damper. Many real-world systems can be approximated as a dual mass-spring-damper system, from a pair of wobbly desks stacked one on top of the other to a two-segmented robotic arm.

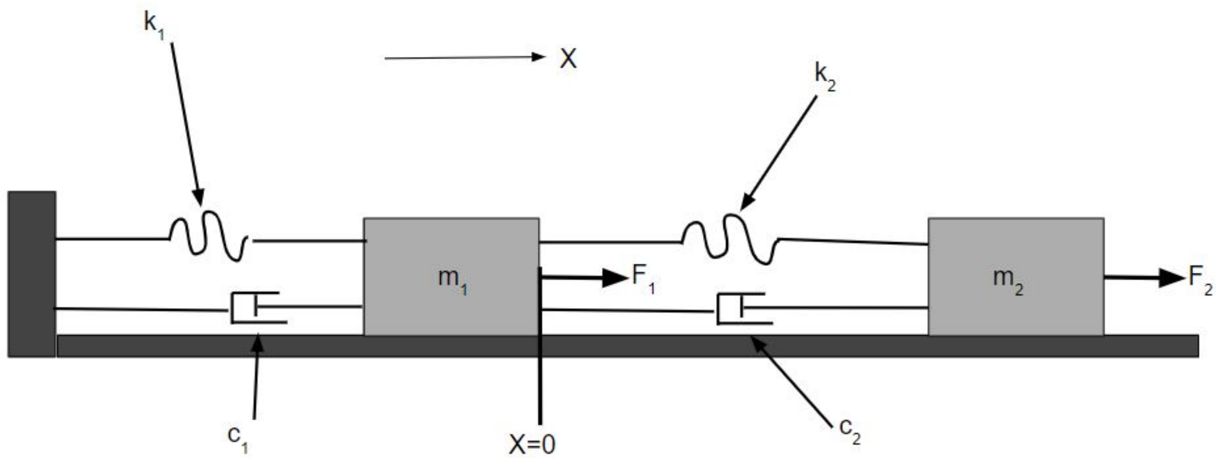

Figure 1: A dual mass-spring-damper system.

The force equation for a single mass-spring-damper is

$$(1) \qquad ma + cv + kx = 0$$

Which can be rewritten as

$$(2) \qquad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Which, as it includes the 0th, 1st, and 2nd derivatives of position x with respect to time t, is a differential equation.

Methods:

In order to understand the motion of the system, a free body diagram must be drawn for each of the two blocks.

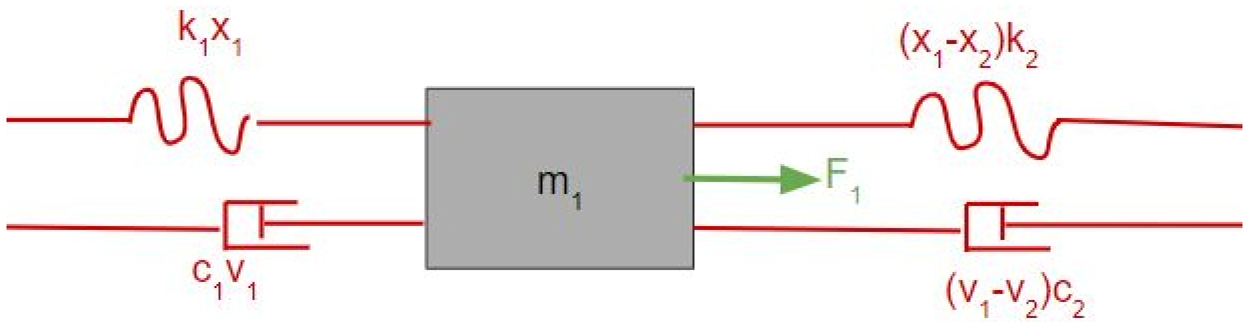

Figure 2: A free body diagram of B_1. Forces acting positively on the block are shown in green, while forces acting negatively on the block are shown in red.

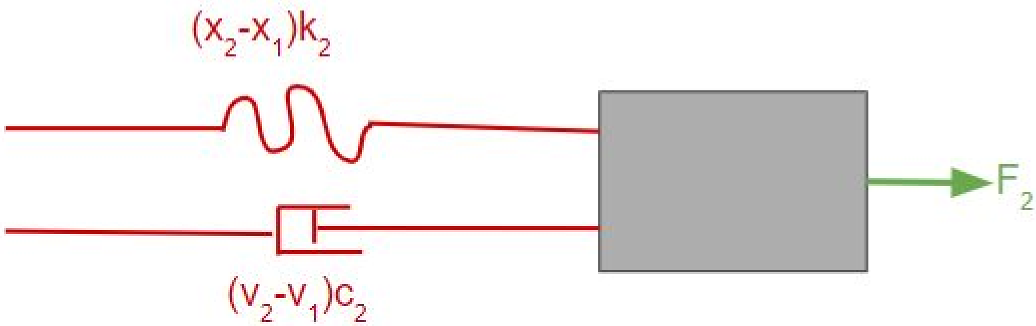

Figure 3: A free body diagram of B_2. Forces acting positively on the block are shown in green, while forces acting negatively on the block are shown in red.

Using these free body diagrams, we can see that the forces acting on Block 1 can be summarized as

$$(3) \qquad \sum F_{B1} = (-x_1k_1)-c_1v_1-(x_1-x_2)k_2-(v_1-v_2)c_2+F_1$$

And the forces on Block 2 can be summarized as

$$(4) \qquad \sum F_{B2} = (-(x_2-x_1)k_2)-c_2(v_2-v_1)+F_2$$

Where x represents horizontal position, v represents velocity (the first derivative of position with respect to time), k represents the spring constants, and c represents the damping coefficients. F_{B1} and F_{B2} are the summations of all forces acting on block 1 and block 2, respectively, while F_1 and F_2 are the external forces acting on those blocks.

Since we now know the forces acting on the block as a function of its position and velocity, we can determine its acceleration by

$$(5) \qquad a = \frac{F}{m}$$

Which itself derives from Newton's second law, which states that the total force acting on an object is equal to its mass multiplied by its acceleration. The acceleration of a block can then be integrated over a small time step to determine the change in the block's velocity, and the velocity of a block can be integrated to determine the change in its position. Since the blocks' initial positions and velocities are known, their positions at any time can thereby be determined numerically.

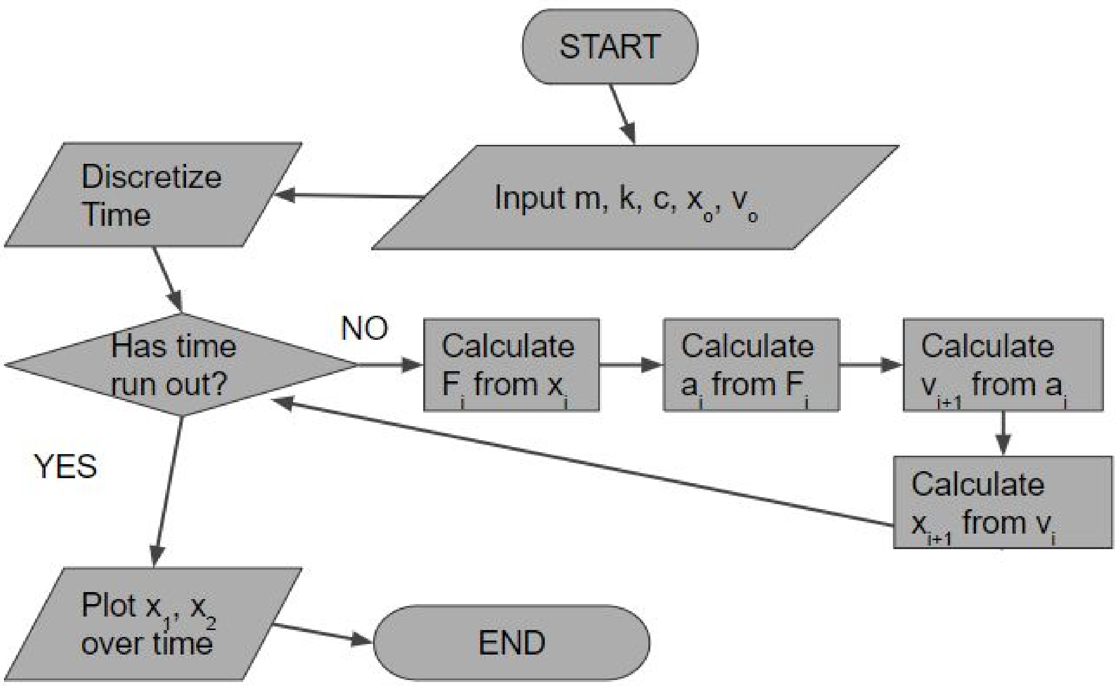

Figure 4: A flowchart describing how a program would be structured to calculate the positions of blocks 1 and 2 over time.

Script:

```
% Problem Statement:
% ========================================================================
% This script calculates the positions over time of the masses in a double
% mass-spring-damper system.

clc; clear all; close all;

% Defining Variables:
% ========================================================================
```

```
% Discretized Time
dt = 0.01; t = 0:dt:15; % s: time step and discretized time
Nt = length(t); % number of discretized timesteps

% System dimensions:
m = [10 5]; % kg: masses m1, m2
k = [10 5]; % N/m: spring constants k1, k2
c = [.05 1]; % Ns/m: damping coefficients c1, c2
L = 2; % m: resting distance between m1 and m2
x0 = [-1 3]; % m: initial positions of m1 and m2
v0 = [0 0]; % m/s: initial velocities of m1 and m2
F = zeros(Nt,2);

% Solution Variables:
x = zeros(Nt,2); % m: positions x2, x2 of m1 and m2
x(1,:) = x0; % initial x
v = zeros(Nt,2); % m/s: velocities v1, v2 of m1 and m2
v(1,:) = v0; % initial v
a = zeros(Nt,2); % m/s^2: accelerations a1, a2 of m1 and m2

% Solution:
% ===========================================================================

for i = 1:(Nt-1)
  F1 = (-x(i,1)*k(1)-c(1)*v(i,1)-(x(i,1)-x(i,2))*k(2)...
     -(v(i,1)-v(i,2))*c(2))+F(i,1); % N: Force acting on m1
  F2 = (-(x(i,2)-x(i,1))*k(2))-c(2)*(v(i,2)-v(i,1))+F(i,2); % N: Force acting on m2
  a(i,:) = [F1/m(1), F2/m(2)]; % m/s^2: a=F/m (Newton's Law)
  v(i+1,:) = v(i,:)+(a(i,:).*dt); % m/s: integrate a over dt for dv
  x(i+1,:) = x(i,:)+(v(i,:).*dt); % m: integrate v over dt for dx
end % end for

% Results:
% ===========================================================================
figure(1)
  plot(t,x(:,1)); hold on
  plot(t,(x(:,2)+L));
  title({'Position of Dual Mass-Spring-Damper System over Time';...
     'With Small m1, k1, c1'});
  xlabel('Time t (seconds)');
  ylabel('Position x (meters)');
  legend('mass 1', 'mass 2');
  grid on;
```

Results:

As intended, the script produced a plot of the absolute positions of the two blocks over time. The blocks are positioned two meters apart naturally.

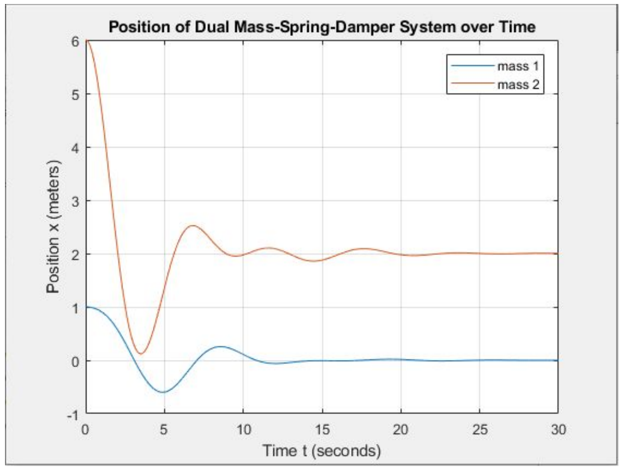

Figure 5: Position of blocks 1 and 2 over time, where m_1=30kg, m_2=5kg, k_1=20N/m, k_2=5N/m, c_1=20Ns/m, and c_2=1Ns/m. The initial displacement of block 1 was 1m and the initial displacement of block 2 was 2m; both blocks started from rest. dt = 0.01 seconds.

Discussion:

Both halves of the system are visibly underdamped. The most notable impact of one block's motion on the other is around time t=7s, when block 2 crosses its resting point from the positive direction but is quickly thrown back by spring 2 due to the proximity of block 1.

Validation:

The validation of these results came in two parts. First, each of the two mass-spring-damper assemblies was inspected in relative isolation from the other and compared to a single mass-spring-damper system. The code for the one-block system had been developed and tested previously in class, and its accuracy is not in question. Both of the two halves were found to work flawlessly in isolation (Figure 6).

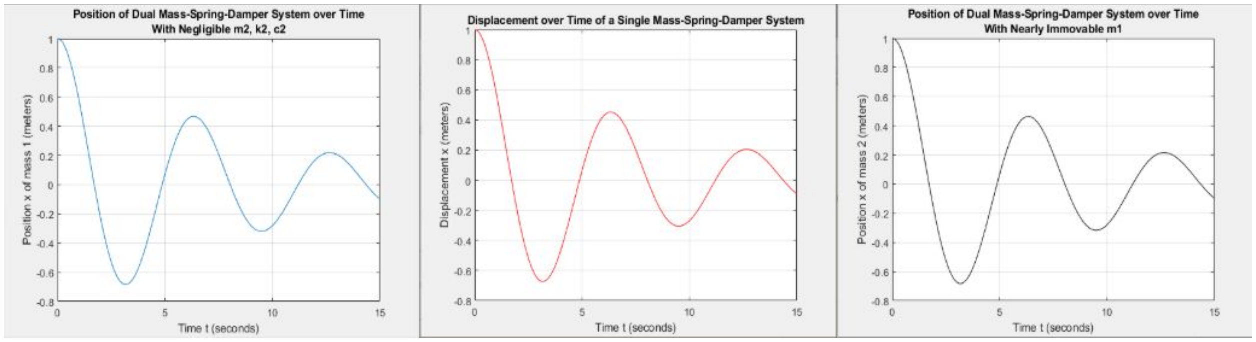

Figure 6: The two blocks of the dual mass-spring-damper system in relative isolation from each other (left and right) compared to a working result from a single mass-spring-damper system (center).

Next, the system as a whole was compared to an analytical solution (Figures 7, 8, and 9). For these trials, $m_1=m_2=1$kg, $k_1=6$N/m, $k_2=4$N/m, and $c_1=c_2=0$Ns/m. Both blocks started from rest with no external forcing. The numerical approximation was found to be extremely accurate when dt was small (0.0001 seconds), but diverged quickly when dt was increased to 0.01 seconds.

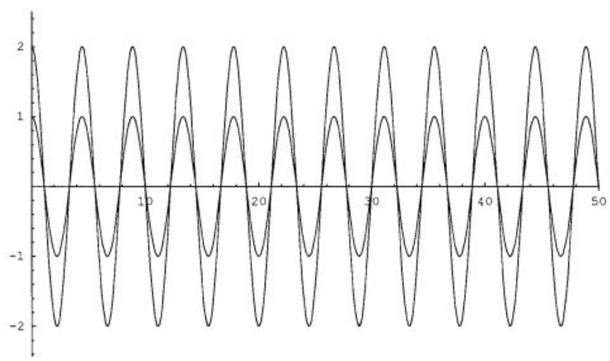

Figure 7: Analytical solution for the displacement over time of both blocks in a dual mass-spring-damper system. The system can be observed to experience approximately 11.25 oscillations over 50 seconds (Fay and Graham 2003).

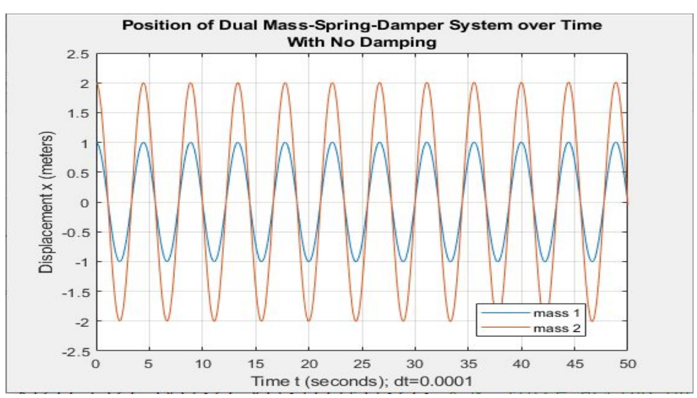

Figure 8: A numerical approximation of the displacement over time of both blocks in a dual mass-spring-damper system with a time step of 0.0001 seconds. The system can be observed to be extremely accurate when compared to the analytical solution in Figure 7.

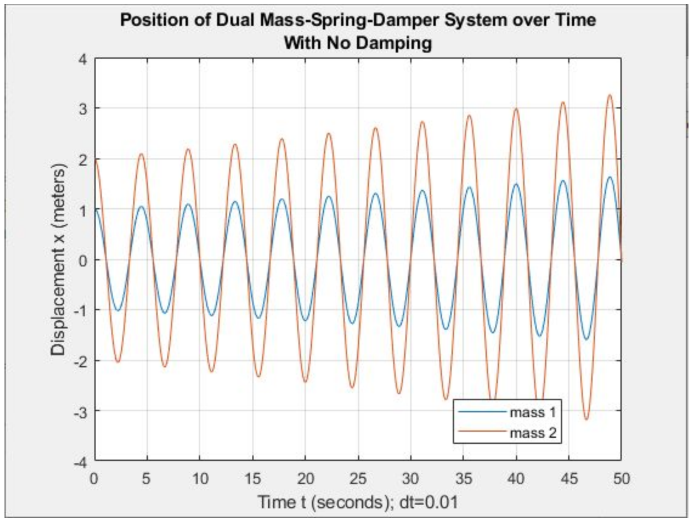

Figure 9: A numerical approximation of the displacement over time of both blocks in a dual mass-spring-damper system with a time step of 0.01 seconds. The system can be observed to diverge from the analytical solution (Figure 7) as time increases.

Conclusion:

The numerical solution is extremely accurate when the time step dt is small. Despite this, it remains incapable of detecting collisions, and so can output unrealistic solutions. Any solution in which the paths of Block 1 and Block 2 cross over each other are likely to be inconsistent with the real world, in which no two objects can occupy the same space simultaneously.

References:

Fay, Temple H, and Sarah Duncan Graham. 2003. "Coupled Spring Equations". *International Journal Of Mathematical Education In Science And Technology* 34 (1): 65-79. http://math.oregonstate.edu/~gibsonn/Teaching/MTH323-010S15/Supplements/coupled_spring.pdf.